Linear Beam Position Control in Optical Waveguides

By JAMES C. DALY

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This paper studies the effect of an active control system that introduces corrective transverse lens displacements on the propagation of beams in optical waveguides. The induced displacement of the nth *lens is a linear function of the beam displacement at the* $(n + 1)$th *lens. Beam displacements from the ideal design position are caused by transverse displacements of the lenses and sensors from their design positions. Expressions have been obtained for:*

(*i*) *The conditions for spatial and temporal stability.*

(*ii*) *The rms beam displacement resulting from uncorrelated displacements of the lenses and sensors.*

(*iii*) *The beam displacement caused by sinusoidal displacements of the lenses and sensors of arbitrary spatial frequency.*

(*iv*) *The spatial rate of return to the guide axis of a beam injected into the guide off axis.*

We also show that the positions of beams, other than the sensed beam, are controlled by the system and that the system can simultaneously guide many different sensed beams. We calculate an upper limit on the probability that the beam will not be contained within a given aperture.

I. INTRODUCTION

The ability of a beam waveguide to guide optical frequency electromagnetic energy is severely limited by the tolerances held on the transverse positions of the lenses. Hirano and Fukatsu[1] have shown that uncorrelated transverse lens displacements will cause the expected deviation of the beam from the guide axis to grow as the square root of the number of lenses through which it has passed. Steier[2] has shown that random longitudinal lens displacements and variations in the focal lengths of the lenses couple with random transverse lens displacements to cause the expected value of the rms beam displacement to grow exponentially with the number of lenses through which it has passed.

This paper describes the performance of linear self-aligning beam waveguides that are subjected to transverse disturbances of the lens and sensor positions. It is quite likely that nonlinear control systems may turn out to be more effective or practical in practice but an understanding of linear systems will facilitate the more involved analysis of nonlinear control systems. The idea of using active guiding media for optical communications was proposed long ago by R. Kompfner[3] and L. U. Kibler,[4] and has been discussed by E. A. J. Marcatili.[5] The improvement that can be obtained with self-aligning beam waveguides has been demonstrated experimentally and with computer simulations by Christian, Goubau, and Mink.[6]

In the guiding systems considered in this paper, the lenses are physically moved an amount that is determined by sensing the position of the beam in the guide.

The axis of the guide is defined as the line joining the centers of the beam sensors. In general in an actual installation this axis will not be straight.

The displacement of paraxial beams from the guide axis is a linear function of the transverse lens displacements. An important consequence of this linearity is that the superposition principle can be applied to the beam displacement. Incremental displacements of the lenses produce incremental displacements of the beam. These increments are independent of any bias positions resulting from an intentional bend. Therefore, when we study the effects of imperfections in a straight guide, the results apply equally well to the effect of imperfections in a curved guide. For example, in studying the effect of the control mechanism on the propagation of a beam through a bend we would proceed as follows:

(*i*) Specify the guide axis. This is the path of propagation for the beam. The sensors are located along this axis.

(*ii*) Find the proper set of lens positions that will cause the beam to propagate along this axis. In an analysis these are calculated. In an actual installation these positions would be determined by moving the lenses until the beam is aligned along the sensors.

(*iii*) Determine the effect of imperfections by assuming that the lenses and sensors are displaced from their proper positions. The consequence of the linearity of the system is that the beam displacement resulting from these imperfections is the same as it would be if the guide axis were a straight line.

This paper emphasizes the case where the design axis for the lens centers and sensors are straight lines and coincident with a reference line. The beam is injected into the guide along the reference line and the center of the first lens of the guide lies on this line.

The optical waveguide consists of a sequence of identical positive lenses, of focal length, f, and spacing, d, as shown in Figure 1. In Figure 1, g_n, s_n, and r_n are the sensor, lens, and beam displacements at the n^{th} lens, respectively. We shall consider paraxial ray propagation in two dimensions for simplicity. It has been shown that the three-dimensional problem can be split into two independent two-dimensional problems[1] and that the center of the beam follows a path described by a paraxial ray.[7, 8]

When the guide is subjected to disturbances in time, the transverse displacement of the n^{th} lens relative to the reference line is given by $s(t)_n$. Similarly, the displacement of the beam center at the n^{th} lens from the reference line is given by $r(t)_n$. Using geometric optics,[1] and considering the guide to be time-dependent, results in the following difference equation for the paraxial beam in the guide

$$r(t)_{n+2} - (2 \cos \omega_0 d) r(t)_{n+1} + r(t)_n = (d/f) s(t)_{n+1} \tag{1}$$

where $\cos \omega_0 d = 1 - d/2f$.

In any lens waveguide there will be transverse displacements of the lenses, owing to the impossibility of holding exact tolerances or the movement of the earth after the guide has been constructed. These lens displacements are unavoidable. They are denoted by N_n. Here we study the effect of a superimposed corrective lens displace-

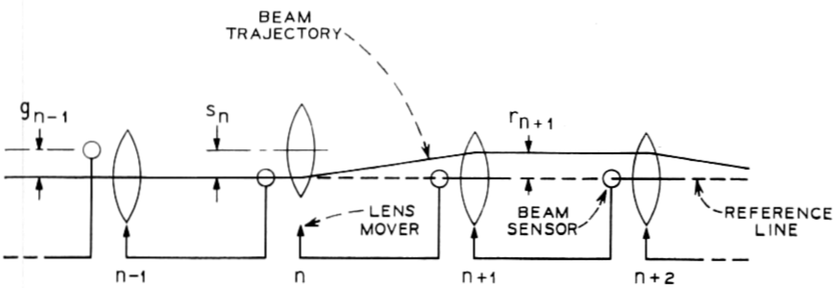

Fig. 1 — The beam waveguide and schematic diagram of the feedback system.

ment, δ_n, on the performance of the beam waveguide. The total displacement of the n^{th} lens is the sum of N_n and δ_n

$$s_n = N_n + \delta_n . \tag{2}$$

For most of this paper, the response of the system in time is determined only in the steady-state when the disturbances are either step functions in time or at low enough temporal frequencies to be considered constant functions independent of time. These conditions are relaxed and general time functions allowed in Section 2.2 where temporal stability is considered and in Section 2.4 where the high gain case is considered.

Although the beam control is achieved here by inducing transverse lens displacements, the results can be applied to other beam control mechanisms such as mirrors and prisms, when such mechanisms can be interpreted as an equivalent transverse displacement of the lens.

II. THE SYSTEM

Figure 2 is a schematic representation of a mechanism for lens position control.

The beam position is sensed at each lens. This beam position signal is used to control the transverse position of the preceding lens. The

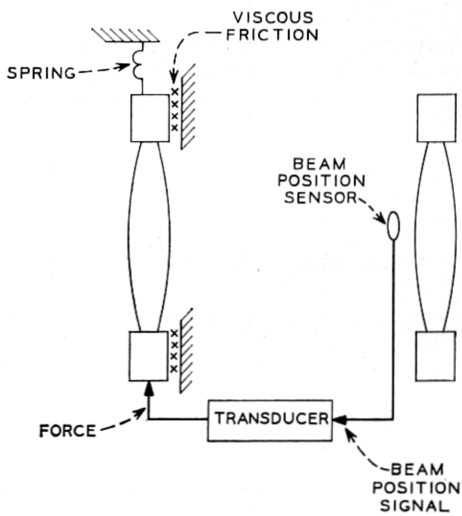

Fig. 2 — Schematic diagram of a mechanical device for lens positioning.

difference between the beam and sensor positions at the nth lens we call the error in the beam position. The corrective displacement of the nth lens, $\delta_n(t)$, is a function of time and the error in the beam position at the $(n+1)$th lens.

$$\delta_n = F(e_{n+1}, t) \tag{3}$$

where e_{n+1} is the position of the beam relative to the sensor at the $(n+1)$th lens. The form that the function $F(e_{n+1}, t)$ takes will depend on the properties of the control mechanism.

In order to study the stability of the beam in the guide it is convenient to transform equation (3) into the temporal frequency domain. Assuming that the control mechanism can be represented by a transfer function $H_c(s)$, the Laplace transform of the corrective displacement of the nth lens for a system at rest at time zero can be written:

$$\delta_n(s) = H_c(s)E_{n+1}(s) \tag{4}$$

where s, $\delta_n(s)$, and $E_{n+1}(s)$ are the complex temporal frequency, the Laplace transform of the corrective displacement of the n^{th} lens, and the Laplace transform of the error in the beam position at the $(n+1)^{\text{th}}$ lens.

In general the transfer function will have the following form

$$H_c(s) = \frac{K(s\tau_a + 1)(s\tau_b + 1) \cdots}{s^N(s\tau_1 + 1)(s\tau_2 + 1)(s\tau_3 + 1) \cdots}. \tag{5}$$

It is customary to classify systems in terms of the number, N, of factors of the form $1/s$ which appear in the transfer function. This number indicates the number of integrations performed by the components. For example: if $N = 0$ the system is referred to as a type 0 system, if $N = 1$ the system is referred to as a type 1 system, and so on.

The number of integrations determines some important properties of the steady state response. Since $s \to 0$ as $t \to \infty$, in the steady state equation (5) yields

$$H_c(s)\big|_{s\to 0} = A = \frac{K}{s^N} \tag{6}$$

when $s \to 0$. There are two possibilities for A. A is either equal to a real constant K (type 0 system) or A goes to infinity (type one or more system). We consider both of these possibilities in the following paragraphs.

For the control mechanism shown in Figure 2, $H_c(s)$, the control mechanism transfer function, will be of the form

$$H_c(s) = \frac{K_o}{Ms^2 + Bs + K_s} \tag{7}$$

where K_o is the gain, K_s is the spring constant, M is the mass of the lens, and B is a friction constant. The steady-state value of $H_c(s) = A = K_o/K_s$. This system is reduced to a type 1 system by removing the spring, that is, letting $K_s = 0$.

2.1 *The General Description*

The properties of the linear active guiding medium in time and space can be studied using a combination of Laplace and Z transform techniques.[9–11] Laplace transforms are used to transform from the time domain to the temporal frequency domain. Quantities that depend on position along the guide, given by the lens number n, are transformed into quantities that depend on axial spatial frequencies using Z transform techniques.

Taking the Laplace transform of equation (1) results in the following recursion equation for the Laplace transform of the beam displacement.

$$[1 - (d/f)H_c(s)]R(s)_{n+2} - (2 \cos \omega_o d)R(s)_{n+1} + R(s)_n = (d/f)\frac{N_{n+1}}{s} - (d/f)H_c(s)\frac{G_{n+2}}{s} \tag{8}$$

where $R(s)_n$, N_n, and G_n are the Laplace transform of the beam displacement at the nth lens, and the constant displacement of the nth sensor, respectively. Taking the Z transform of equation (8) results in the following equation when the boundary conditions are zero, that is, the beam is injected into the guide on axis and the first lens of the guide is on axis

$$R(s, z)\{[1 - (d/f)H_c(s)]z^2 - (2 \cos \omega_o d)z + 1\} = z(d/f)\frac{N(z)}{s} - z^2(d/f)H_c(s)\frac{G(z)}{s} \tag{9}$$

where $R(s,z)$ is the Z transform of the Laplace transform of the beam position, and $N(z)$ and $G(z)$ are the Z transforms of the constant lens and sensor displacements, respectively. Rewriting equation (9) and introducing the transfer function $H(s,z)$ results in the following expression.

$$R(s, z) = H(s, z)[N(z) - zH_c(s)G(z)]/s \tag{10}$$

where

$$H(s, z) = \frac{z(d/f)}{[1 - (d/f)H_c(s)]z^2 - (2 \cos \omega_o d)z + 1}. \tag{11}$$

Equation (10) is a simple relationship between functions that contain the information about the performance of the guide. The transfer functions $H(s,z)$ and $H_c(s)$ contain the properties of the guide such as d/f and the characteristics of the control mechanism. The function $R(s,z)$ contains all the information that is necessary in principle to determine the position of the beam at any lens as a function of time. $N(z)$ and $G(z)$ are determined from the constant lens and sensor displacements.

When $r_n(t)$ can be considered a constant function of time denoted by r_n, then $R(s,z)$ is the Z transform of r_n/s. Multiplying equation (10) by s and letting the temporal frequency go to zero yields $R(z)$, the Z transform of the steady-state beam position

$$R(z) = H(z)[N(z) - AzG(z)] \tag{12}$$

where $H(z)$ is equal to the value of $H(s,z)$ for s equal to zero.

2.2 *Temporal Stability*

A function is defined as stable in time if it is bounded for all positive values of time. The position of the beam in the guide will be stable in time if the poles of the Laplace transform of the beam position are in the left half of the complex temporal frequency plane (s plane). The Laplace transform of the beam position is obtained by taking the inverse Z transform of equation (10). The following convolution summation[10] is a general expression for the inverse Z transform and gives the Laplace transform of the beam position.

$$R_n(s) = \frac{1}{s} \sum_{m=0}^{n} H_m(s)[N_{n-m} - H_c(s)G_{n-m+1}] \tag{13}$$

where $H_m(s)$ is the inverse Z transform of $H(s,z)$. Assuming that the transfer function $H_c(s)$ (that describes the control mechanism) has no poles in the right half of the s plane, the only source of right half plane poles would be in the function $H_m(s)$. In order to obtain an expression for $H_m(s)$ it is convenient to factor the denominator of equation (11) and rewrite the expression for $H(s,z)$. z_1 and z_2 are the two roots of the denominator in equation (11) and are given by the following expression

$$z_{1 \text{ and } 2} = \frac{\cos \omega_o d \pm [\cos^2 \omega_o d - 1 + (d/f)H_c(s)]^{\frac{1}{2}}}{1 - (d/f)H_c(s)}. \tag{14}$$

Therefore

$$H(s, z) = \left[\frac{d/f}{1 - (d/f)H_c(s)}\right] \frac{z}{(z - z_1)(z - z_2)}. \tag{15}$$

The inverse Z transform of equation (15) can be found in tables and is given by

$$H_m(s) = \frac{(d/2f)[z_1^m - z_2^m]}{[\cos^2 \omega_o d - 1 + (d/f)H_c(s)]^{\frac{1}{2}}}. \tag{16}$$

A sufficient condition for temporal stability is that the poles $H_m(s)$ are in the left half of the temporal frequency plane. The s plane poles of $H_m(s)$ are the roots of the following two equations

$$1 - (d/f)H_c(s) = 0 \tag{17}$$

and

$$\sin^2 \omega_o d - (d/f)H_c(s) = 0. \tag{18}$$

Equation (17) is the characteristic equation of a single sensor lens loop. Therefore if a single section of guide is stable the roots of equation (17) will be in the left half of the s plane. Equation (18) is more restrictive. For example, if a single section becomes unstable when the gain, K in equation (5), is increased beyond a value K_u, it follows from equation (18) that when many sections are tied together to form the optical waveguide, the value of K at which the system becomes unstable is reduced by $\sin^2 \omega_o d$.

When the $H_c(s)$ is given by equation (7), the system is stable if

$$A < 1 - d/4f \tag{19}$$

where $A = K_o/K_s$.

2.3 *Spatial Stability*

A guide will be defined as spatially stable if the beam remains within a bounded region around the guide axis for all values of n. A sufficient condition for the steady-state time solution to be spatially stable is that the poles of $H(z)$ are inside the unit circle in the Z plane. $H(z)$ can be obtained by setting $s = 0$ in equation (11).

$$H(z) = \frac{z(d/f)}{[1 - (d/f)A]z^2 - (2 \cos \omega_o d)z + 1}. \tag{20}$$

Beginning with equations (19) and (20), using standard techniques it can be shown that for the type 0 system described by equation (7) only a negative value of A will satisfy the conditions for both temporal and spatial stability. For type 1 or higher systems $A \to \infty$ and the poles of $H(z)$ approach the origin of the Z plane assuring spatial stability.

2.4 *The High Gain Case*

By high gain we mean large magnitude of $H_c(s)$. This is of particular importance, since the performance of the system improves when the magnitude of $H_c(s)$ becomes large. When the magnitude of $H_c(s)$ is large, equation (11) becomes

$$H(s, z) \approx \frac{-1}{zH_c(s)}. \tag{21}$$

Using the above approximation for $H(s,z)$ and relaxing the restriction that the disturbances are constant functions of time and substituting into equation (10) the general expressions $N(s,z)$ and $G(s,z)$ for the Z transforms of the Laplace transforms of the lens and sensor displacements yields

$$R(s, z) - G(s, z) \approx -\frac{N(s, z)}{zH_c(s)}. \tag{22}$$

Division of a Z transform by z corresponds to subtracting one from the index of the inverse transform. The inverse Z transform of equation (22) is

$$R_n(s) - G_n(s) \approx -\frac{N_{n-1}(s)}{H_c(s)} \tag{23}$$

when $H_c(s)$ is large the Laplace transforms of the beam and sensor positions will be approximately equal. It follows that the functions in the time domain will also be approximately equal. In the steady-state, when the disturbances are independent of time, the displacement of the beam from the guide axis is

$$r_n - g_n = -\frac{N_{n-1}}{A}. \tag{24}$$

For a type 1 or more system $A \to \infty$ and in the steady-state the beam follows the sensors.

2.5 *The Propagation of Unsensed Beams in the Guide*

An unsensed beam will differ from the sensed beam primarily because the active guiding mechanism will not respond to deviations of the unsensed beam from the guide axis. The unsensed beam may also be injected into the guide off axis and follow an entirely different trajectory in the guide than does the sensed beam.

The effect of the corrective displacements on beams other than the beam that is sensed by the control system, can be seen by noticing the form of the total lens displacement. The Z transform of the total lens displacement, $S(z)$, is the Z transform of the constant lens displacement, $N(z)$ plus the Z transform of the corrective lens displacement $\delta(z) = Az[R(z) - G(z)]$,

$$S(z) = N(z) + Az[R(z) - G(z)]. \tag{25}$$

Substituting equation (12) into equation (25), the Z transform of the total lens displacement becomes

$$S(z) = \frac{z^2 - (2 \cos \omega_o d)z + 1}{(1 - Ad/f)z^2 - (2 \cos \omega_o d)z + 1} [N(z) - AzG(z)]. \tag{26}$$

Let P_n be the displacement of an unsensed beam propagating in the guide with the boundary conditions P_{in} and P'_{in} just to left of the first lens which is assumed to be on the reference line. The difference equation that describes the propagation of any beam sensed or unsensed traveling in either direction through the guide is equation (1) rewritten here to describe the unsensed beam.

$$P_{n+2} - (2 \cos \omega_0 d)P_{n+1} + P_n = (d/f)s_{n+1} . \tag{27}$$

Operating on equation (27) using Z transform techniques[9–11] yields the following expression for the Z transform of the position of the unsensed beam

$$P(z) = \frac{z(d/f)S(z) + [z^2 - z]P_{in} + zdP'_{in}}{z^2 - (2 \cos \omega_o d)z + 1}. \tag{28}$$

The substitution of equation (26) and (20) into equation (28) yields the following expression for the Z transform of the displacement of the unsensed beam.

$$P(z) = H(z)[N(z) - AzG(z)] + \frac{(z^2 - z)P_{in} + zdP'_{in}}{z^2 - (2 \cos \omega_o d)z + 1}. \tag{29}$$

The displacement of the unsensed beam, P_n, the inverse Z transform of (29), is the superposition of two components. The first is represented by the first two terms in equation (29). A comparison of these two terms with equation (12) shows that this component of the displacement is identical to the displacement of the sensed beam. Comparison of the remaining terms of equation (29) with equation (28) shows that they are identical to the Z transform of the displacement of a beam traveling in a guide with zero lens displacement, $S(z) = 0$.

The second component is identical to the displacement of an unsensed beam injected off axis into a perfectly straight guide. The total displacement of the unsensed beam is the sum of these two components. The unsensed beam will oscillate about the position of the sensed beam just as a beam injected off axis into a perfectly straight guide would oscillate about the guide axis. This result differs from equation (28) of Reference 5, which predicts a growth in the displacement of the unsensed beam as it propagates. In Reference 5 substitution shows equation (28) to be inconsistent with equation (22).

For stability, the sensed beam must travel in the direction of increasing n, but an unsensed beam may travel in either direction.

2.6 *The Propagation of Many Sensed Beams in the Guide*

Where there are many beams propagating in the positive direction of the guide, the signal obtained from the sensors is the average displacement of all the beams. The Z transform of the beam position signal is

$$D(z) = \langle P(z) \rangle - G(z). \tag{30}$$

Where $D(z)$, $\langle P(z) \rangle$, and $G(z)$ are the Z transforms of the position signal, the average beam displacement, and the sensor displacement, respectively. Different beams are injected into the guide differently and follow different trajectories through the guide. When the Z transform of the total lens displacement is $S(z)$, the Z transform of any beam propagating in the guide is given by equation (28). The Z transform of the average displacement is given by the average of equation (28) over all beams.

$$\langle P(z) \rangle = \frac{zd/fS(z) + [z^2 - z]\langle P_{in} \rangle + zd\langle P'_{in} \rangle}{z^2 - (2 \cos \omega_o d)z + 1}. \tag{31}$$

Where $\langle P_{in} \rangle$ and $\langle P'_{in} \rangle$ are the average displacement and slope of the beams just before propagating through the zeroth lens. It follows from equation (31) that the average displacement of the beams in the guide is the same as the displacement of a beam that is injected into the guide with slope and displacement equal to $\langle P'_{in} \rangle$ and $\langle P_{in} \rangle$, respectively. Therefore, the control system will respond to the average beam position signal and control the average of the beam displacements just as it would respond to one beam propagating in the guide.

2.7 *The Response of a Beam to Waves of Displacements*

As an example of how the system might perform consider the following particular type of disturbance. Both lenses and sensors are displaced simultaneously. The constant transverse displacements, D_n, of the lens and sensor assembly relative to the straight reference line form an axial wave of spatial frequency ω, and are described by the following equation:

$$D_n = Be^{i\omega nd} \tag{32}$$

then

$$D(z) = \frac{zB}{z - e^{i\omega d}} \tag{33}$$

where B is the amplitude of the wave, $\omega = 2\pi/\lambda$, and λ is the spatial wavelength. When n is large enough so that boundary effects can be neglected, the Z transform of the displacement of the beam from the sensor is obtained by using equation (20) and letting $D(z) = N(z) = G(z)$ in equation (12).

$$R(z) - D(z) = -\frac{(z-1)^2}{(1 - Ad/f)z^2 - 2\cos\omega_o dz + 1} D(z). \tag{34}$$

It follows[10] from equation (34) that at any given spatial frequency, ω, the ratio of the amplitude of the wave of the beam's displacement from the lens sensor assembly, to the amplitude of the wave of the lens sensor assembly's displacement from the axis is given by

$$\left| \frac{r_n - D_n}{D_n} \right| = \frac{|1 - \cos\omega d|}{|\cos\omega d - \cos\omega_o d - A(d/2f)e^{i\omega d}|}. \tag{35}$$

The ratio given in equation (35) is plotted in Figure 3, for various real values of A, as a function of the spatial phase, ωd. It follows from the sampling theorem that the largest spatial phase possible between lenses is $\omega d = \pi$.

Figure 4 is a plot of the ratio of the amplitude of the beam displacement to the lens displacement when the sensors remain on axis and the

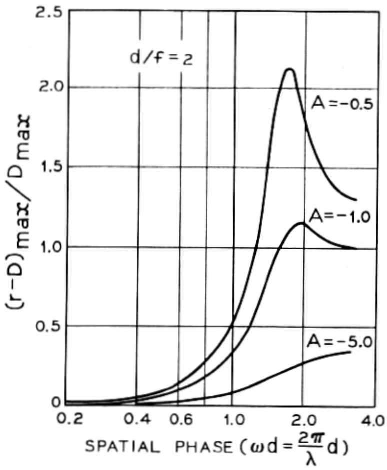

Fig. 3 — Spatial frequency response of the beam position relative to the sensor when the lenses and sensors are disturbed simultaneously.

lens displacements N_n form a wave of axial spatial frequency ω. For this case, the ratio of the amplitude of the beam displacement wave to the amplitude of the lens displacement wave is

$$\left| \frac{r_n}{N_n} \right| = \frac{d/2f}{\left| \cos \omega d - \cos \omega_o d - A(d/2f)e^{i\omega d} \right|}. \tag{36}$$

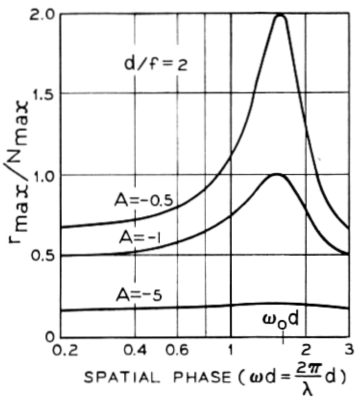

Fig. 4 — Spatial frequency response of the beam position relative to the sensor when sensors are undisturbed and the lenses are disturbed.

2.8 *The Response of a Beam to Random Transverse Displacements*

Again assuming that the entire lens sensor assembly moves together, the spatial response of the beam to any general set of displacements, D_n, is given by the inverse Z transform of equation (34). The displacement of the beam from the sensor is the following convolution summation

$$r_n - D_n = -D_n + \sum_{m=1}^{n} (h_{n-m} - Ah_{n-m+1})D_m \tag{37}$$

where h_n is the inverse Z transform of the transfer function $H(z)$, given in equation (20).

$$h_n = \frac{d/f}{(1 - Ad/f)^{(n+1/2)}} \frac{\sin n\omega_2 d}{\sin \omega_2 d} \tag{38}$$

where $\cos \omega_2 d = (1 - d/2f)/(1 - Ad/f)^{1/2}$. We wish to find the expected value of the square of the beam displacement, from the sensor, $\langle (r_n - D_n)^2 \rangle$. For uncorrelated displacements, the expected value of the product of any two displacements is given by

$$\langle D_n D_m \rangle = \begin{cases} 0 & \text{if} \quad n \neq m \\ \sigma_L^2 & \text{if} \quad n = m \end{cases} \tag{39}$$

where σ_L is the rms lens and sensor displacement. The expected value of the square of the beam position can be written*

$$\langle (r_n - D_n)^2 \rangle = \sigma_L^2 \left[\frac{1 + Ad/f}{1 - Ad/f} + \sum_{m=0}^{n-1} (h_m - Ah_{m+1})^2 \right]. \tag{40}$$

This summation has been evaluated for large n and plotted in Figure 5 for $d/f = 1, 2,$ and 3 as a function of A. When the magnitude of A is large compared with one, the ratio of the rms beam to rms lens and sensor displacement approaches $1/|A|$.

2.9 *The Ability of a Short Section of Stabilized Guide to Return an Initially Off-Axis Beam to the Optical Axis*

The preceding sections are concerned with the effect of lens displacements on a beam that is injected into the guide with zero slope

* Equation (40) can be obtained using equation (37) with the following considerations. (*i*) The displacements, D_n, are uncorrelated. (*ii*) The expected value of a sum is the sum of the expected values. (*iii*) The expected value of the product of a deterministic function and a random variable is the product of the deterministic function and the expected value of the random variable. (*iv*) The zeroth lens and sensor are on axis.

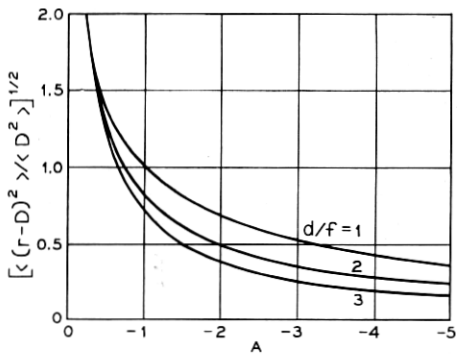

Fig. 5 — The statistical response of beam position relative to the sensor when lenses and sensors are disturbed simultaneously.

and coincident with the guide axis. This is equivalent to saying that the boundary conditions of the difference equation are zero.

When a beam is injected into a straight feedback controlled guide off axis and the lens displacements are zero the Z transform of the beam position is a function of the boundary conditions and is determined using standard techniques.[10]

$$R(z) = \frac{[(1 - Ad/f)z^2 - (2 \cos \omega_o d)z]r_0 + (1 - Ad/f)r_1}{(1 - Ad/f)z^2 - (2 \cos \omega_o d)z + 1} \tag{41}$$

where r_0 and r_1 are the beam displacement at the zeroth and first lenses, respectively. The inverse Z transform of equation (41) can be found in tables and is equal to the following

$$r_n = (1 - Ad/f)^{-n/2} (B \sin \omega_2 nd + C \cos \omega_2 nd) \tag{42}$$

where $\cos \omega_2 d = (1-d/2f)/(1-Ad/f)^{1/2}$ and the constants B and C chosen so that equation (42) will satisfy the boundary conditions.

Equation (42) shows that when a beam is injected into the feedback controlled guide off axis the maximum possible deviation from the axis decreases by a factor of $(1-Ad/f)^{-1/2}$ at each lens. For example, a beam injected parallel to the axis, one centimeter off axis, into a stabilized guide four lenses long with $Ad/f = -9$, will leave the guide with a maximum displacement from the axis that is less than 0.1 mm.

2.10 *The Probability That the Beam Will Not be Contained Within a Given Aperture*

An upper limit on the probability of the beam exceeding a threshold can be calculated using Chebyshev's inequality.[12] The probability of the magnitude of the beam deviation from the axis of the sensors exceeding a value, T, at any given lens is less than the mean squared deviation divided by T^2

$$P(| r_n - g_n | \geqq T) \leqq \frac{\langle (r_n - g_n)^2 \rangle}{T^2}. \tag{43}$$

If the magnitude of "A" is much larger than one, it follows from equation (24) that

$$\langle (r_n - g_n)^2 \rangle = \frac{\sigma_L^2}{A^2} \tag{44}$$

where σ_L is the rms disturbance of the lens position. Equation (44) is quite general and holds for deterministic as well as random lens displacements.

III. CONCLUSIONS

A beam waveguide using the linear beam control system described here can be made stable in time and space by proper choice of the control mechanism.

If many beams are propagating in the guide it is only necessary that the control system interact with one or more beams propagating in the positive direction for the guide to be stabilized for all beams.

When the magnitude of A, the low frequency value of $H_c(s)$ is large, the error in the beam position varies inversely with A.

It follows from equation (24) that, if A is large, the probability that the deviation of the beam from the axis of the sensors will exceed a value, T, at any given lens is equal to the probability that the displacement of the preceding lens will exceed a value of T multiplied by A. If $A = -100$, and $T = 1$, and if the lens displacements have a standard deviation of 0.1 and a gaussian amplitude distribution, then the probability that the deviation of the beam from the guide axis will exceed one is less than the vanishingly small value of 10^{-10^5}. A more conservative number, 10^{-6}, for this probability, that holds for all amplitude distributions, is obtained using Chebyshev's inequality.

There is no growth in the rms deviation of the beam from the guide axis as the beam propagates down the stabilized guide. Also, since the effects of guide imperfections, for example, transverse lens displacements, do not build up, the construction tolerances on the guide can be relaxed. It is possible to contain the beam within a narrow region of the guide axis, which permits minimizing the guide aperture.

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